# Notes on Morse Theory and Handle Decompositions

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# 1. Morse Theory

Morse theory studies a smooth manifold through real-valued functions defined on it. By examining how such a function behaves at certain points, we can uncover topological properties and structure of the manifold. For example, the identity function on $\mathbb{R}$ is unbounded. On the other hand, every continuous function $S^1 \to \mathbb{R}$ is bounded because $S^1$ is compact. Therefore $\mathbb{R}$ and $S^1$ cannot be homeomorphic.

## 1.1. Critical Points

Let $M$ be a smooth $n$-manifold and let $f : M \to \mathbb{R}$ be smooth. At each $p \in M$ the derivative of $f$ is a linear map

$$df_p : T_p M \to \mathbb{R}$$

**Definition 1.1.1**: A point $p \in M$ is a critical point of $f$ if $df_p = 0$. A number $c \in \mathbb{R}$ is a critical value if $c = f(p)$ for some critical point $p$. Points which are not critical are called regular points, and values which are not critical values are called regular values.

In local coordinates $(x_1, ..., x_n)$ around $p$, the condition $df_p = 0$ is equivalent to

$$\frac{\partial f}{\partial x_1}(p) = ... = \frac{\partial f}{\partial x_n}(p) = 0$$

Thus this definition agrees with the familiar multivariable-calculus definition. Geometrically, when $M$ is drawn as a surface and $f$ is a height function, a critical point is a point at which the tangent plane is horizontal.

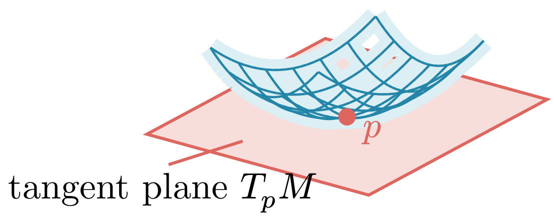


Figure 1: A critical point of a height function. The tangent plane is horizontal at $p$.

The derivative tells us that the first-order behavior vanishes at a critical point. To understand what happens next, we examine the second derivatives.

Suppose $p$ is a critical point of a twice continuously differentiable function $f : M \to \mathbb{R}$. In local coordinates around $p$, the Hessian of $f$ at $p$ is the $n \times n$ matrix $H_f(p)$ whose $(i, j)$'th entry is

$$\left(H_f(p)\right)_{ij} = \frac{\partial^2 f}{\partial x_i \partial x_j}(p)$$

By Clairaut's theorem, the mixed partial derivatives are equal, so the Hessian is symmetric. Although the matrix representing the Hessian changes when we change coordinates, at a critical point it changes by congruence. Consequently, whether it is invertible and the numbers of its positive and negative eigenvalues do not depend on the chosen coordinates.

**Definition 1.1.2**: A critical point $p$ of $f$ is nondegenerate if its Hessian is invertible. Equivalently, its determinant satisfies the following condition. $\det H_f(p) \neq 0$ A critical point whose Hessian is not invertible is called degenerate.

The Hessian describes how the function bends near a critical point. A positive direction is one in which $f$ initially rises away from the point, while a negative direction is one in which it initially falls. A zero eigenvalue leaves a direction in which the quadratic approximation gives no information. This is why nondegenerate critical points have especially well controlled local behavior.

## 1.2. The Morse Lemma

Since the Hessian is a real symmetric matrix, the spectral theorem implies that all of its eigenvalues are real. At a nondegenerate critical point none of these eigenvalues is zero.

**Definition 1.2.1**: The index of a nondegenerate critical point $p$, denoted $\mathrm{ind}(p)$, is the number of negative eigenvalues of $H_f(p)$, counted with multiplicity.

The index counts the independent directions in which the function decreases as we move away from the critical point. On a surface, an index 0 point is a local minimum, an index 1 point is a saddle, and an index 2 point is a local maximum.

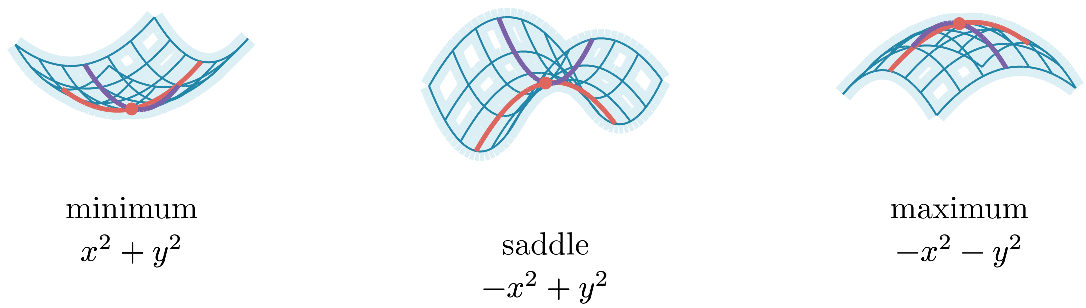


Figure 2: The three possible indices of a nondegenerate critical point on a surface.

**Theorem 1.2.1 (The Morse Lemma)**: Let $p$ be a nondegenerate critical point of a smooth function $f : M \to \mathbb{R}$, and let $\lambda = \mathrm{ind}(p)$. There are local coordinates $(x_1, ..., x_n)$ centered at $p$. In these coordinates, $f$ has the following form.

$$f(x) = f(p) - x_1^2 - ... - x_\lambda^2 + x_{\lambda+1}^2 + ... + x_n^2$$

The Morse lemma is stronger than an ordinary second-order Taylor approximation: in suitable coordinates there are no higher-order error terms at all. Thus, near a nondegenerate critical point, the function has a form determined entirely by the index of the critical point.

**Proof:**
Without loss of generality, choose a coordinate chart $\varphi : U \to V \subseteq \mathbb{R}^n$ such that $\varphi(p) = 0$. In these coordinates, $f$ is represented by the function $F = f \circ \varphi^{-1}$. Replacing $F$ by $F - F(0)$ allows us to assume $F(0) = 0$. This shift by a constant does not change the first derivatives, the Hessian, the critical point, or its index. For readability, we write $f$ for this coordinate representation throughout the remainder of the proof. We prove the result via induction on the dimension $n$.

First suppose the dimension is 1. Taylor's theorem to second order gives

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2}x^2 + R_2(x)$$

The integral form of the second-order Taylor remainder is

$$R_2(x) = \frac{1}{2}\int_0^x (x-s)^2 f'''(s)\,\mathrm{d}s$$

Using the change of variables $s = tx$, so that $\mathrm{d}s = x\,\mathrm{d}t$, we obtain $R_2(x) = x^3 r(x)$, where

$$r(x) = \frac{1}{2}\int_0^1 (1-t)^2 f'''(tx)\,\mathrm{d}t$$

Since $f(0) = 0$ and $f'(0) = 0$, the constant and first-order terms vanish. Therefore

$$f(x) = x^2\left(\frac{f''(0)}{2} + xr(x)\right) = x^2 h(x)$$

where $h(x) = \frac{f''(0)}{2} + xr(x)$ is smooth and $h(0) = \frac{f''(0)}{2}$. Nondegeneracy implies that $h(0) \neq 0$. After restricting to a smaller neighborhood, $h$ has a constant sign $\varepsilon \in \{-1, 1\}$. Define

$$y = x\sqrt{|h(x)|}$$

The derivative of this coordinate change at 0 is $\sqrt{|h(0)|} \neq 0$. By the inverse function theorem, $y$ is a valid local coordinate. In this coordinate

$$f = \varepsilon y^2$$

This proves the one-dimensional case.

Now suppose the result holds in dimension $n-1$. Since the Hessian of $f$ at 0 is invertible, an invertible linear change of coordinates allows us to assume

$$\frac{\partial^2 f}{\partial x_1^2}(0) \neq 0$$

Let $x' = (x_2, ..., x_n)$. Applying the implicit function theorem to the equation

$$\frac{\partial f}{\partial x_1}(x_1, x') = 0$$

We have there is a smooth $g$ defined near 0 such that

$$\frac{\partial f}{\partial x_1}(g(x'), x') = 0$$

Let $u = x_1 - g(x')$. For each fixed $x'$, the function $u \mapsto f(u + g(x'), x')$ has vanishing first derivative at $u = 0$. The one-dimensional factorization used above, now with $x'$ treated as a parameter, gives a smooth function $h$ such that

$$f(u + g(x'), x') = f(g(x'), x') + u^2 h(u, x')$$

Moreover,

$$h(0, 0) = \frac{1}{2}\frac{\partial^2 f}{\partial x_1^2}(0) \neq 0$$

After shrinking the neighborhood again, $h$ has a constant sign $\varepsilon$. The change of coordinate

$$y_1 = u\sqrt{|h(u, x')|}$$

is locally invertible at the origin. If we define

$$f_1(x') = f(g(x'), x')$$

then the preceding factorization becomes

$$f = \varepsilon y_1^2 + f_1(x')$$

The origin is a critical point of $f_1$. The coordinate changes above transform the original Hessian into a block diagonal matrix whose first block corresponds to $\varepsilon y_1^2$ and whose remaining block is the Hessian of $f_1$. Since the original Hessian is invertible, the Hessian of $f_1$ is also invertible. Its number of negative eigenvalues is $\lambda$ when $\varepsilon = 1$, and $\lambda - 1$ when $\varepsilon = -1$.

The induction hypothesis applies to $f_1$. It gives coordinates $(y_2, ..., y_n)$ in which $f_1$ is a sum of positive and negative squares. Including the square $\varepsilon y_1^2$ produces exactly $\lambda$ negative squares and $n - \lambda$ positive squares. Therefore

$$f(x) = f(p) - y_1^2 - ... - y_\lambda^2 + y_{\lambda+1}^2 + ... + y_n^2$$

which proves the result. ■

**Theorem 1.2.2**: A nondegenerate critical point of a function is isolated.

**Proof:**
Let $p$ be a nondegenerate critical point of $f : M \to \mathbb{R}$. Choose a coordinate chart $\varphi : U \to V \subseteq \mathbb{R}^n$ such that $\varphi(p) = 0$, and let $F = f \circ \varphi^{-1}$ be the coordinate representation of $f$. Define the map $G : V \to \mathbb{R}^n$ by

$$G(x) = \left(\frac{\partial F}{\partial x_1}(x), ..., \frac{\partial F}{\partial x_n}(x)\right)$$

A point $x \in V$ represents a critical point of $f$ exactly when $G(x) = 0$. Since $p$ is critical, $G(0) = 0$.

The derivative of $G$ at the origin is

$$dG_0 = H_F(0)$$

This matrix is invertible because $p$ is nondegenerate. By the inverse function theorem, there is a neighborhood $W \subseteq V$ of the origin on which $G$ is a diffeomorphism onto its image. In particular, $G$ is injective on $W$. Since $G(0) = 0$, no other point of $W$ can also be mapped to 0. Therefore $p$ is the only critical point represented in $W$, so it is isolated. ■

**Definition 1.2.2**: We say that a smooth function $f : M \to \mathbb{R}$ is a Morse function if all its critical points are nondegenerate.

One notes that the subsequent development of Morse theory relies only on the $C^2$ condition. Thus if Morse functions were defined as $C^2$ functions then the following theory would remain identical.

**Theorem 1.2.3**: Every smooth, compact manifold $M$ admits a Morse function, and Morse functions form an open and dense set in the space of all smooth functions on $M$.

**Proof:**
Let $n = \dim M$. The first jet bundle $J^1(M, \mathbb{R})$ consists of triples

$$(x, a, \ell), x \in M, a \in \mathbb{R}, \ell \in T_x^* M$$

The first jet of a smooth function $f : M \to \mathbb{R}$ is the map

$$j^1 f : M \to J^1(M, \mathbb{R}), j^1 f(x) = (x, f(x), df_x)$$

Thus the first jet records the point, the value of the function, and its first derivative at that point. Consider the subset

$$W = \{(x, a, \ell) \in J^1(M, \mathbb{R}) : \ell = 0\}$$

This is a closed smooth submanifold of $J^1(M,\mathbb{R})$. In local coordinates, $J^1(M,\mathbb{R})$ has coordinates

$$(x_1, ..., x_n, a, \xi_1, ..., \xi_n)$$

and $W$ is defined by $\xi_1 = ... = \xi_n = 0$. Hence $W$ has codimension $n$.

The Jet Transversality Theorem states that, for any smooth submanifold $W \subseteq J^1(M,\mathbb{R})$, the set of smooth functions $f$ for which $j^1 f$ intersects $W$ transversely is dense in $C^\infty(M,\mathbb{R})$. When $M$ is compact and $W$ is closed, this set is also open.

Since $j^1 f(x) \in W$ exactly when $df_x = 0$, the map $j^1 f$ meets $W$ precisely at the critical points of $f$. Let $x$ be such a point and choose local coordinates around it. In these coordinates,

$$j^1 f = (x_1, ..., x_n, f, \partial_1 f, ..., \partial_n f)$$

The directions normal to $W$ are the final $n$ coordinates. The component of $d(j^1 f)_x$ in these directions is therefore the linear map represented by

$$H_f(x)$$

Consequently, $j^1 f$ is transverse to $W$ at $x$ if and only if $H_f(x)$ has full rank. Since the Hessian is an $n \times n$ matrix, this is equivalent to its being invertible. Therefore $j^1 f$ is transverse to $W$ if and only if every critical point of $f$ is nondegenerate, which is precisely the condition that $f$ is a Morse function.

Jet transversality now gives us that the Morse functions form an open and dense subset of $C^\infty(M,\mathbb{R})$. In particular, this set is nonempty, so $M$ admits a Morse function. ■

## 1.3. Handle Decompositions

In the following sections we study how to decompose manifolds into handles. For simplicity we will only study connected manifolds noting that any disconnected manifold partitions into connected components.

**Definition 1.3.1**: A $k$-handle in dimension $n$ is $D^k \times D^{n-k}$.

One notes that every $n$-dimensional handle is inherently diffeomorphic to $D^n$. The index $k$ does not change the internal shape of the handle itself. Instead it specifies exactly which portion of the handle's boundary is glued to the existing manifold.

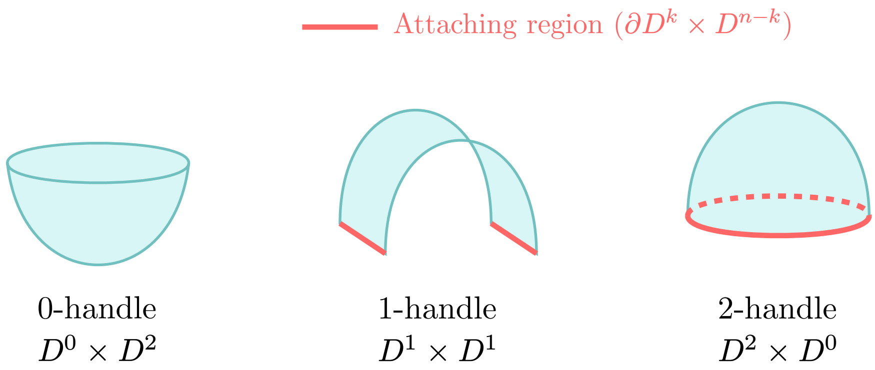


Figure 3: The two dimensional handles embedded in $\mathbb{R}^3$

**Definition 1.3.2**: For $c \in \mathbb{R}$, the sublevel set of $f : M \to \mathbb{R}$ at $c$ is $M_c = f^{-1}((-\infty, c])$

**Lemma 1.3.1 (Regular Interval)**: Let $f : M \to \mathbb{R}$ be smooth, where $M$ is compact. If $f$ has no critical points with values in $[a, b]$, then $M_a$ and $M_b$ are diffeomorphic.

We will assume this lemma to be true.

Thus a sublevel set can change only when its boundary passes a critical point.

**Theorem 1.3.1 (The Handle Attachment Theorem)**: Let $p$ be an index $k$ critical point of a Morse function $f : M \to \mathbb{R}$, and let $c = f(p)$. Suppose that $p$ is the only critical point with value in $[c - \varepsilon, c + \varepsilon]$. Then $M_{c+\varepsilon}$ is diffeomorphic to $M_{c-\varepsilon}$ with a $k$-handle attached along its boundary.

**Proof:**
By the Morse lemma, there are coordinates $(u_1, ..., u_k, v_1, ..., v_{n-k})$ centered at $p$ in which

$$f(u, v) = c - \|u\|^2 + \|v\|^2$$

Choose $\varepsilon > 0$ small enough that these coordinates are defined on a neighborhood containing

$$H = \{(u, v) : \|u\|^2 \leq 2\varepsilon, \|v\|^2 \leq \varepsilon\} \cong \overline{B}_{\mathbb{R}^k}\left(0, \sqrt{2\varepsilon}\right) \times \overline{B}_{\mathbb{R}^{n-k}}(0, \sqrt{\varepsilon}) \cong D^k \times D^{n-k}$$

Every point of $H$ satisfies

$$f(u, v) \leq c + \varepsilon$$

so $H$ lies in $M_{c+\varepsilon}$. On the part of its boundary where $\|u\|^2 = 2\varepsilon$, we have

$$f(u,v) = c - 2\varepsilon + \|v\|^2 \leq c - \varepsilon$$

Thus this part of the boundary already lies in $M_{c-\varepsilon}$. This set is precisely the attaching region

$$\partial \overline{B}_{\mathbb{R}^k}\left(0, \sqrt{2\varepsilon}\right) \times \overline{B}_{\mathbb{R}^{n-k}}\left(0, \sqrt{\varepsilon}\right) \cong \partial D^k \times D^{n-k}$$

We have now described the change inside the coordinate neighborhood of $p$. Outside this neighborhood, there are no critical points with values between $c - \varepsilon$ and $c + \varepsilon$. By the regular interval lemma, this portion of the sublevel set does not change topologically. Therefore the only topological change occurs inside the coordinate neighborhood, where $H$ is attached along its attaching region. Hence

$$M_{c+\varepsilon} \cong M_{c-\varepsilon} \cup_{\partial D^k \times D^{n-k}} \left(D^k \times D^{n-k}\right)$$

so passing the critical point attaches a $k$-handle. ■

**Lemma 1.3.2**: A Morse function $f : M \to \mathbb{R}$ on compact $M$ has only a finite number of critical points.

**Proof:**
Let $C$ be the set of critical points of $f$. We first show that $C$ is closed. If $q$ is not critical, then in any local coordinate system around $q$, at least one first partial derivative of $f$ is nonzero at $q$. By continuity, this partial derivative remains nonzero on a sufficiently small neighborhood of $q$. Every point in this neighborhood is therefore regular, so the set of regular points is open and $C$ is closed.

Since $M$ is compact and $C$ is closed, $C$ is compact. Because $f$ is a Morse function, every point of $C$ is nondegenerate, and nondegenerate critical points are isolated. Thus for every $p \in C$, there is an open neighborhood $U_p$ such that

$$U_p \cap C = \{p\}$$

The collection of all $U_p$ is an open cover of $C$. Compactness gives a finite subcover $U_{p_1}, ..., U_{p_m}$. Each set in this subcover contains only one critical point, so $C = \{p_1, ..., p_m\}$. Therefore $f$ has finitely many critical points. ■

Therefore any compact smooth manifold can be entirely constructed as a finite handle decomposition.

**Lemma 1.3.3 (Handle Cancellation)**: An index $k$ handle can be smoothly canceled with an index $k + 1$ or $k - 1$ handle, provided their attaching boundaries intersect transversely in a single point.

We will only prove this lemma in the case of surfaces.

**Proof:**
Let $h^0$ be the 0-handle and let $h^1 = D^1 \times D^1$ be the 1-handle. The attaching region of $h^1$ consists of the two intervals $\partial D^1 \times D^1$. By assumption, one of these intervals is attached to $\partial h^0$, while the other is attached to the surface constructed before this pair is added.

The union of the disk $h^0$ with the band $h^1$ is again a disk. From the viewpoint of the earlier surface, this disk is attached along a single interval in its boundary. Such an attachment only extends the boundary and does not change the diffeomorphism type of the surface. Hence $h^0$ and $h^1$ can be removed from the decomposition.

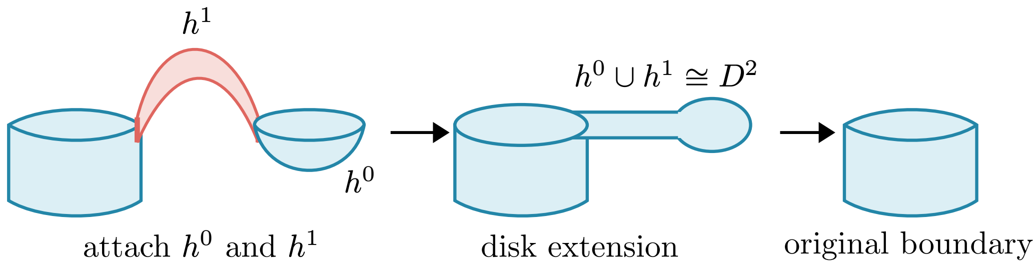


Figure 4: The union $h^0 \cup h^1$ is a disk-shaped extension which can be smoothed back into the original boundary.

Now consider a cancelling 1-handle and 2-handle. If $f$ is the Morse function producing the decomposition, then $-f$ is also a Morse function. On a surface, replacing $f$ by $-f$ changes an index $k$ critical point into an index $2-k$ critical point. The original 2-handle therefore becomes a 0-handle, while the original 1-handle remains a 1-handle. The order of attachment is reversed, so the pair becomes a cancelling 0-handle and 1-handle for $-f$. It cancels by the argument above. Therefore the original 1-handle and 2-handle also cancel. ■

## 1.4. Classification of Curves

**Theorem 1.4.1**: Any connected 1-manifold without boundary is diffeomorphic to either the circle $S^1$ or the real line $\mathbb{R}$.

**Proof:**
Choose a countable, locally finite handle decomposition of $M$.

In dimension 1, both types of handles are intervals. A 0-handle is

$$D^0 \times D^1 \cong [-1, 1]$$

and a 1-handle is

$$D^1 \times D^0 \cong [-1, 1]$$

The two endpoints of a 1-handle form its attaching region. Since $M$ has no boundary, every endpoint in the handle decomposition must be joined to an endpoint of another handle.

Begin inside a 0-handle and move toward one of its endpoints. At this endpoint we enter an attached 1-handle. Crossing the 1-handle takes us to a 0-handle, and we can continue through its other endpoint into the next 1-handle. Repeating this process gives an alternating chain of 0-handles and 1-handles.

This chain cannot branch because every handle is an interval with exactly two endpoints. One endpoint is the point through which we entered and the other is the point through which we continue. Moreover, connectedness implies that every handle belongs to this chain. Otherwise the remaining handles would form a separate connected component of $M$.

Suppose first that $M$ is compact. Local finiteness and compactness imply that there are only finitely many handles. Since $M$ has no boundary, the chain cannot stop. It must therefore return to the handle from which it began. Thus all the handles form a single finite closed chain, which is diffeomorphic to the circle $S^1$.

Now suppose that $M$ is noncompact. The chain cannot return to a handle it has already visited. If it did, it would form a finite closed chain with no free endpoint through which another handle could attach. Connectedness would then imply that this closed chain is all of $M$, contradicting noncompactness. Since $M$ has no boundary, the chain cannot stop in either direction. Consequently its handles can be indexed as

$$..., h_{-2}, h_{-1}, h_0, h_1, h_2, ...$$

with each $h_i$ joined to $h_{i+1}$ at one endpoint. We can identify each handle smoothly with $[i, i+1]$ so that the identifications agree at their shared endpoints. The entire chain is therefore diffeomorphic to $\mathbb{R}$. Hence every connected 1-manifold without boundary is diffeomorphic to either $S^1$ or $\mathbb{R}$. ■

# 2. 2-Manifolds

We now use handle decompositions to classify all compact and connected surfaces.

## 2.1. Handle Decomposition of Surfaces

Let $\Sigma$ be our compact and connected surface and let $f : \Sigma \to \mathbb{R}$ be a Morse function.

**Lemma 2.1.1**: Any compact and connected surface $\Sigma$ has a handle decomposition with exactly one 0-handle, $k$ 1-handles, and one 2-handle.

**Proof:**

Begin with any 0-handle. If there is another 0-handle, connectedness of $\Sigma$ guarantees that some 1-handle joins the part already constructed to a part containing another 0-handle. This 1-handle can be arranged to meet the boundary of that 0-handle once, so the pair cancels. Repeating this process leaves exactly one 0-handle.

Now replace $f$ by $-f$. The 2-handles for $f$ become 0-handles for $-f$, while the 1-handles remain 1-handles. The same argument cancels all but one of them. Returning to $f$, the resulting decomposition has exactly one 0-handle, some number $k$ of 1-handles, and exactly one 2-handle. ■

**Definition 2.1.1**: A Morse function $f : M \to \mathbb{R}$ is called self-indexing if $f(p) = \operatorname{ind}(p)$ for every critical point $p \in M$.

In the case of surfaces, a self-indexing Morse function places all index 1 critical points at the same level. Thus all the 1-handles attach to the boundary of the 0-handle along disjoint attaching regions. We may therefore attach them in any order without changing the resulting surface.

**Lemma 2.1.2**: Let $M$ be a smooth, compact manifold. Then $M$ admits a self-indexing Morse function.

We will assume this lemma to be true.

## 2.2. Connected Sums

To classify all compact and connected surfaces, we break them down into simpler building blocks.

**Definition 2.2.1**: Given two $n$-dimensional manifolds $M_1$ and $M_2$ their connected sum denoted as $M_1 \# M_2$ is formed by removing the interior of a closed $n$ -disk from each manifold and identifying the resulting $S^{n-1}$ boundaries via a diffeomorphism. The construction is expressed by the following formula. $M_1 \# M_2 = (M_1 \setminus \operatorname{int}(D^n)) \cup_{S^{n-1}} (M_2 \setminus \operatorname{int}(D^n))$

Notice that the $n$-sphere $S^n$ acts as the identity element for the connected sum operation. Taking the connected sum of a manifold $M$ with $S^n$ involves removing the interior of a disk $D^n$ from each manifold. Notice that $S^n \setminus \operatorname{int}(D^n) \cong D^n$. Gluing this $D^n$ back onto the boundary of $M \setminus \operatorname{int}(D^n)$ perfectly reconstructs our original manifold $M$. Therefore, $M \# S^n \equiv M$.

**Definition 2.2.2**: A compact and connected manifold $\Sigma \ncong S^2$ is called prime if it cannot be decomposed as a non-trivial connected sum. That is if $\Sigma \cong M_1 \# M_2$, then either $M_1 \cong S^n$ or $M_2 \cong S^n$.

Every compact and connected surface can be decomposed into a connected sum of prime surfaces. Indeed, if the surface is not prime, split it as a non-trivial connected sum and repeat this process with every factor which is not prime. This process must terminate because the surface has a finite handle decomposition, and each non-trivial factor contains at least one of its 1-handles. We eventually obtain a finite connected sum in which every factor is prime.

We now investigate the prime surfaces.

## 2.3. Prime Surfaces

**Lemma 2.3.1**: Let $\Sigma$ be a prime surface. There does not exist a nonempty proper subset $A$ of the 1-handles such that the boundary $\partial(D^2 \cup A)$ is connected.

**Proof:**
Among all handle decompositions of $\Sigma$, choose one with the smallest possible number of handles. Such a decomposition exists because $\Sigma$ has a finite handle decomposition and the number of handles is a nonnegative integer. In particular, no further handle cancellation can reduce this decomposition.

Suppose for the sake of contradiction that such a proper subset $A$ exists. Because $\partial(D^2 \cup A)$ is a compact and connected 1-manifold it must be diffeomorphic to the circle $S^1$.

Let $B$ be the remaining set of 1-handles. Because $\partial(D^2 \cup A) \cong S^1$, we can attach a 2-handle to this boundary to close it off, forming a compact surface $\Sigma_A = D^2 \cup A \cup D^2$.

The handles in $B$ and the final 2-handle of $\Sigma$ attach to this same $S^1$ boundary and so we can similarly define $\Sigma_B$. Therefore our original surface $\Sigma$ can be viewed as the connected sum of the surface formed by $A$ and the surface formed by $B$

$$\begin{aligned}\Sigma &\cong D^2 \cup A \cup B \cup D^2 \\ &\cong (D^2 \cup A \cup D^2) \mathbin{\#} (D^2 \cup B \cup D^2) \\ &\cong \Sigma_A \mathbin{\#} \Sigma_B\end{aligned}$$

Since $A$ is nonempty and proper, neither $A$ nor $B$ is empty. Thus both $\Sigma_A$ and $\Sigma_B$ contain at least one 1-handle. If either surface were diffeomorphic to $S^2$, its punctured surface would be a disk. We could then replace all the handles in that punctured piece by a single disk, producing a handle decomposition of $\Sigma$ with fewer handles. This contradicts our choice of a decomposition with the smallest possible number of handles. Therefore neither $\Sigma_A$ nor $\Sigma_B$ is diffeomorphic to $S^2$.

This means $\Sigma$ is a connected sum of two non-trivial manifolds which contradicts our assumption that $\Sigma$ is a prime manifold. Therefore no such proper subset $A$ can exist. ■

We can now determine exactly how many 1-handles a prime manifold must have by analyzing how 1-handles affect the boundary.

**Lemma 2.3.2**: Every prime surface is diffeomorphic to the real projective plane $\mathbb{RP}^2$ (non-orientable) or the torus $\mathbb{T}^2$ (orientable).

**Proof:**
Assume $\Sigma$ is a prime surface. Then $\Sigma$ must have at least one 1-handle. In general the boundary of our surface at any intermediate stage of adding 1-handles is a compact 1-manifold which is a disjoint union of circles $S^1 \cup S^1 \cup \ldots$

We split this into two cases based on orientability

Case 1: (The Non-Orientable Case) If we attach a 1-handle such that it reverses orientation (adding a twist, locally forming a Mobius band), the boundary of the 0-handle combined with this 1-handle remains a single connected circle $S^1$. By our previous lemma, a proper subset of 1-handles cannot yield a connected boundary. Since a single twisted 1-handle yields a connected boundary, it cannot be a proper subset and thus it must be the entire set of 1-handles and thus $k_1 = 1$. Attaching the single 2-handle to this boundary yields our non-orientable prime surface. To identify this manifold we see that $\mathbb{RP}^2$ can be represented by a square $D^2$ with the boundary identifications shown below. In this model, the disk we remove is represented by the identified top and bottom portions of the square. Removing it gets rid of the identifications along the top and bottom, while the two vertical sides remain identified in opposite directions. What remains is the rectangle model of a Mobius band.

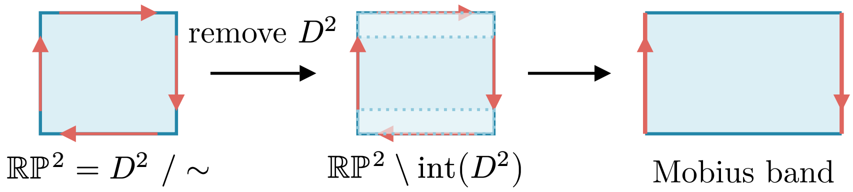


Figure 5: The punctured projective plane is diffeomorphic to a Mobius band.

$$\mathbb{RP}^2 \setminus \text{int}(D^2) \cong \text{Mobius band}$$

The union of the 0-handle and twisted 1-handle is also a Mobius band. Attaching the final 2-handle is therefore the same as gluing a disk back onto $\mathbb{RP}^2 \setminus \text{int}(D^2)$. Hence the resulting surface is $\mathbb{RP}^2$.

Case 2: (The Orientable Case) If we attach a 1-handle such that it preserves orientation, it splits the original $S^1$ boundary of the 0-handle into two disjoint $S^1$ components. To eventually close the surface with a single 2-handle, the final boundary before attaching the 2-handle must be a single $S^1$. Note that adding an orientable 1-handle to the same $S^1$ boundary component splits it into two and adding an orientable 1-handle connecting different $S^1$ boundary components merges them into one.

Since the boundary must eventually recombine into a single $S^1$ so that the 2-handle can attach, at least one of the remaining 1-handles must connect the two different components created by the first 1-handle. Using a self-indexing Morse function, all the 1-handles attach at the same level and may be attached in any order. We may therefore attach this connecting 1-handle immediately after the first one. It merges the two boundary components back into a single $S^1$.

The union of the 0-handle with these two 1-handles now has connected boundary. By the previous lemma, these two handles cannot form a proper subset of all the 1-handles. Therefore they are the entire collection, so $k_1 = 2$. Attaching the 2-handle to the remaining connected boundary yields the torus $\mathbb{T}^2$.

Therefore, the only prime surfaces are $\mathbb{RP}^2$ and $\mathbb{T}^2$. ■

**Lemma 2.3.3**: $\mathbb{T}^2 \# \mathbb{RP}^2 \cong \mathbb{RP}^2 \# \mathbb{RP}^2 \# \mathbb{RP}^2$

**Proof:**
Remove a disk from each surface before forming the connected sum. The punctured torus is a disk with two untwisted 1-handles, while the punctured projective plane is a Mobius band, which is a disk with one twisted 1-handle.

The connected sum is formed by identifying their two boundary circles. Instead of identifying these circles directly, we may glue one boundary circle to each end of the cylinder $S^1 \times D^1$. This produces the same surface because the cylinder merely stretches the region between the two circles.

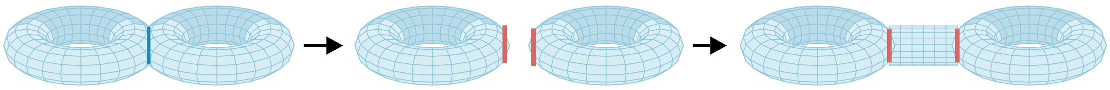


Figure 6: The gluing operation, illustrated with two tori, is unchanged when the product cylinder $S^1 \times D^1$ is inserted between the boundary circles.

We can cut the cylinder into two disks and remove the interior of one of them. The remaining part is

$$D^1 \times D^1 \cong D^2$$

Its attaching region is

$$\partial D^1 \times D^1$$

which consists of two disjoint intervals. One interval is attached to the connected boundary of the punctured torus and the other is attached to the connected boundary of the punctured projective plane.

We may slide each attaching interval along its connected boundary until it lies on the boundary of the corresponding 0-handle. The disk $D^1 \times D^1$ is then a 1-handle joining the two 0-handles. By the Handle Cancellation Lemma, this connecting 1-handle cancels with one of the 0-handles. We are left with a single 0-handle together with the two untwisted 1-handles from the punctured torus and the twisted 1-handle from the punctured projective plane. Thus the punctured connected sum has one 0-handle and three 1-handles. Gluing the disk removed from the cylinder back at the end gives the final 2-handle.

The argument above shows more generally how to combine two minimal handle decompositions. The connecting 1-handle cancels one of the two 0-handles, so the connected sum has a single 0-handle carrying all the 1-handles from both surfaces, followed by a single 2-handle.

We now have a handle decomposition of $\mathbb{T}^2 \# \mathbb{RP}^2$ consisting of one 0-handle, two untwisted 1-handles, one twisted 1-handle, and one 2-handle. We may slide the end of a 1-handle along the boundary without changing the surface. Slide one end of the first untwisted 1-handle through the twisted 1-handle. Passing through the twist reverses that end, so the untwisted 1-handle becomes twisted. Repeating the same move with the other untwisted 1-handle makes it twisted as well. The surface is now represented by a disk with three twisted 1-handles.

The minimal handle decomposition of $\mathbb{RP}^2$ has one 0-handle, one twisted 1-handle, and one 2-handle. Applying the connected sum argument above repeatedly, the connected sum of three copies of $\mathbb{RP}^2$ has one 0-handle, three twisted 1-handles, and one 2-handle. This is exactly the handle decomposition obtained above for $\mathbb{T}^2 \# \mathbb{RP}^2$. Hence $\mathbb{T}^2 \# \mathbb{RP}^2 \cong \mathbb{RP}^2 \# \mathbb{RP}^2 \# \mathbb{RP}^2$. ■

## 2.4. Euler Characteristic

We now characterize all compact and connected surfaces by their orientability and their euler characteristic.

**Definition 2.4.1**: For a manifold $M$ its euler characteristic

$$\chi(M) = \sum (-1)^i k_i$$

where $k_i$ is the number of $i$ handles in the decomposition of $M$.

The above definition seems to depend on the choice of handle decomposition, but it can be shown that the resulting value is independent of this choice. We will assume that the Euler characteristic is well-defined and show that it is a topological invariant.

**Lemma 2.4.1**: If $A \cong B$ then $\chi(A) = \chi(B)$.

**Proof:**
Let $F : A \to B$ be a diffeomorphism and choose a handle decomposition of $A$. An $i$-handle $h$ is diffeomorphic to

$$D^i \times D^{n-i}$$

and its attaching region is

$$\partial D^i \times D^{n-i}$$

Since the restriction of $F$ to $h$ is also a diffeomorphism, $F(h)$ is diffeomorphic to the same product, and $F$ carries the attaching region of $h$ to the corresponding attaching region of $F(h)$. Thus $F(h)$ is again an $i$-handle.

Moreover, $F$ is bijective, so this gives a one-to-one correspondence between the handles in the decomposition of $A$ and their images in $B$. Therefore, if the decomposition of $A$ has $k_i$ handles of index $i$, the resulting decomposition of $B$ has exactly the same number. The two decompositions consequently have the same alternating sum, and hence $\chi(A) = \chi(B)$. ■

We see that orientable surfaces are completely determined by their genus $g$ (the number of holes). Their Euler characteristic is given by $\chi = 2 - 2g$. Non-orientable surfaces are completely determined by their non-orientable genus $k$ (the number of projective planes). Their Euler characteristic is given by $\chi = 2 - k$.

Thus two compact and connected surfaces with the same Euler characteristic are diffeomorphic if they are either both orientable or both non-orientable.

## 2.5. Classification of Surfaces

Now that we have identified the prime surfaces, we can classify all compact and connected surfaces. Every compact and connected surface can be decomposed into a connected sum of prime surfaces. Since $S^2$ acts as the identity for the connected sum ($\Sigma \# S^2 \cong \Sigma$), every compact, connected surface is formed entirely by connecting tori ($\mathbb{T}^2$) and real projective planes ($\mathbb{RP}^2$).

**Theorem 2.5.1**: Every compact and connected surface $\Sigma$ is diffeomorphic to exactly one of the following

(i) A sphere $S^2$.
(ii) A connected sum of $g$ tori: $\mathbb{T}^2 \# \mathbb{T}^2 \# ... \# \mathbb{T}^2$ (for $g \geq 1$).
(iii) A connected sum of $k$ real projective planes: $\mathbb{RP}^2 \# \mathbb{RP}^2 \# ... \# \mathbb{RP}^2$ (for $k \geq 1$).

Notice that our theorem does not include a case that mixes tori and projective planes. This is as $\mathbb{T}^2 \# \mathbb{RP}^2 \cong \mathbb{RP}^2 \# \mathbb{RP}^2 \# \mathbb{RP}^2$. Consequently any mixed connected sum can be turned into a sum of strictly projective planes.

# 3. 3-Manifolds

We now apply Morse theory and handle decompositions to closed, connected, and orientable 3-manifolds.

## 3.1. Handle Decomposition of Solids

**Theorem 3.1.1**: Let $M$ be a closed manifold of odd dimension. Then

$$\chi(M) = 0$$

**Proof:**

Write $\dim(M) = 2m + 1$. Let $f : M \to \mathbb{R}$ be a Morse function and let $k_i$ be the number of critical points of index $i$. The corresponding handle decomposition gives

$$\chi(M) = \sum_{i=0}^{2m+1} (-1)^i k_i$$

Now consider the Morse function $-f$. A critical point of index $i$ for $f$ has index $2m + 1 - i$ for $-f$. Thus the handle decomposition produced by $-f$ has $k_{2m+1-i}$ handles of index $i$. Since the Euler characteristic is independent of the chosen handle decomposition,

$$\begin{aligned}\chi(M) &= \sum_{i=0}^{2m+1} (-1)^i k_{2m+1-i} \\ &= \sum_{j=0}^{2m+1} (-1)^{2m+1-j} k_j \\ &= -\sum_{j=0}^{2m+1} (-1)^j k_j \\ &= -\chi(M)\end{aligned}$$

Therefore $2\chi(M) = 0$, and hence $\chi(M) = 0$. ■

**Lemma 3.1.1**: Every closed, connected, and orientable 3-manifold admits a handle decomposition with exactly one 0-handle and one 3-handle.

**Proof:**

Choose a Morse function on $M$. Since $M$ is closed, its corresponding handle decomposition is finite. Begin with the union of its 0-handles. Each 0-handle is initially a separate connected component, and the 1-handles join these components together.

Handles of index at least 2 cannot join two different connected components. Indeed, the attaching region of a $k$-handle is

$$\partial D^k \times D^{3-k}$$

which is connected when $k \geq 2$, so its image must lie in a single connected component. Since the final manifold $M$ is connected, the 0-handles and 1-handles must already form a connected set.

Represent each 0-handle by a vertex and each 1-handle joining two different 0-handles by an edge. This gives a finite connected graph. Choose a spanning tree in this graph. A spanning tree has a leaf, which corresponds to a 0-handle joined to the rest of the tree by exactly one chosen 1-handle. After sliding any other 1-handle attachments away from this 0-handle, the chosen 1-handle meets it in exactly one attaching disk. The Handle Cancellation Lemma therefore cancels this 0-handle with the chosen 1-handle.

Removing a leaf from a tree leaves a smaller tree. We may therefore repeat the cancellation until only one 0-handle remains.

Now replace the Morse function $f$ by $-f$. Under this reversal, the 3-handles for $f$ become 0-handles for $-f$, while the 2-handles become 1-handles. Applying the same argument to $-f$ leaves exactly one 0-handle for $-f$, and hence exactly one 3-handle for $f$. These cancellations do not change the number of 0-handles already obtained for $f$. Thus $M$ admits a handle decomposition with exactly one 0-handle and one 3-handle. ■

**Lemma 3.1.2**: In a handle decomposition of a closed, connected, and orientable 3-manifold with one 0-handle and one 3-handle, the numbers of 1-handles and 2-handles are equal.

**Proof:**
Indeed, if there are $k_1$ 1-handles and $k_2$ 2-handles, then the handle decomposition gives

$$\chi(M) = 1 - k_1 + k_2 - 1$$

Since $\chi(M) = 0$, we obtain $k_1 = k_2$.

## 3.2. Heegaard Splittings

**Definition 3.2.1**: A handlebody of genus $g$ is a compact 3-manifold obtained by attaching $g$ 1-handles to a single 0-handle. Its boundary is a closed and orientable surface of genus $g$.

**Definition 3.2.2**: A Heegaard splitting of a closed and orientable 3-manifold $M$ is a decomposition

$$M = V_1 \cup_{\Sigma_g} V_2$$

where $V_1$ and $V_2$ are genus-$g$ handlebodies with common boundary

$$\partial V_1 = \Sigma_g = \partial V_2$$

The 0-handle and 1-handles form the first handlebody $V_1$. If the Morse function is replaced by its negative, the 3-handle becomes a 0-handle and the 2-handles become 1-handles. They therefore form the second handlebody $V_2$.

**Theorem 3.2.1**: Every closed, connected, and orientable 3-manifold admits a Heegaard splitting.

**Proof:**
By the previous results, choose a handle decomposition of $M$ with one 0-handle, $g$ 1-handles, $g$ 2-handles, and one 3-handle. Choose a self-indexing Morse function which produces this decomposition. Its critical points of indices 0, 1, 2, and 3 occur at levels 0, 1, 2, and 3, respectively.

Choose a regular value $c \in (1, 2)$ and let $V_1 = M_c$. The sublevel set $V_1$ contains the 0-handle and all the 1-handles, but none of the 2- or 3-handles. It is therefore obtained from a single 0-handle by attaching $g$ 1-handles, so $V_1$ is a handlebody of genus $g$.

Let $V_2$ be the closure of the remaining part of $M$. It contains the 2-handles and the 3-handle. View this piece using the Morse function $-f$. The 3-handle for $f$ becomes a 0-handle for $-f$, and the $g$ 2-handles become $g$ 1-handles. Thus $V_2$ is also a handlebody of genus $g$.

The two pieces meet along the regular level surface

$$\Sigma_g = f^{-1}(\{c\})$$

which is the boundary of both handlebodies. Hence

$$M = V_1 \cup_{\Sigma_g} V_2$$

is a Heegaard splitting of $M$. ■

**Definition 3.2.3**: The Heegaard genus of $M$ is the smallest genus among all Heegaard splittings of $M$.

## 3.3. Lens Spaces

A genus-one handlebody is a solid torus

$$V_i \cong D^2 \times S^1$$

Its boundary is the torus $\mathbb{T}^2$. On $\partial V_i$, let $\mu_i$ denote the meridian, which bounds a disk in $V_i$, and let $\lambda_i$ denote the longitude, which travels around the $S^1$ direction of $V_i$.

Every simple closed curve on $\mathbb{T}^2$ is represented by

$$q\lambda_1 + p\mu_1$$

where $p$ and $q$ are relatively prime.

**Definition 3.3.1**: The lens space $L(q,p)$ is obtained by gluing two solid tori $V_1$ and $V_2$ so that the meridian $\mu_2$ is identified with the curve

$$q\lambda_1 + p\mu_1$$

on $\partial V_1$.

**Theorem 3.3.1**: Every closed and orientable 3-manifold admitting a genus-one Heegaard splitting is diffeomorphic to a lens space $L(q,p)$.

**Proof:**

Let $M = V_1 \cup_\varphi V_2$ be a genus-one Heegaard splitting. Both $V_1$ and $V_2$ are solid tori, and the gluing map is a diffeomorphism

$$\varphi : \partial V_2 \to \partial V_1$$

The meridian $\mu_2$ is a simple closed curve on $\partial V_2$. Its image under $\varphi$ is therefore a simple closed curve on $\partial V_1 \cong \mathbb{T}^2$. With respect to the longitude and meridian of $V_1$, this curve has the form

$$\varphi(\mu_2) = q\lambda_1 + p\mu_1$$

for some integers $p$ and $q$.

To see why $p$ and $q$ are relatively prime, consider the universal cover of the torus as $\mathbb{R}^2/\mathbb{Z}^2$. A lift of the closed curve $\varphi(\mu_2)$ to $\mathbb{R}^2$ begins at some point $x$ and ends at

$$x + (q,p)$$

If $p$ and $q$ had a common divisor $d > 1$, then $(q,p) = d(q',p')$ for some integers $p'$ and $q'$. The curve would travel around the curve represented by $(q',p')$ a total of $d$ times, so it would not be a simple closed curve. Since $\varphi$ is a diffeomorphism and $\mu_2$ is simple, its image $\varphi(\mu_2)$ is also simple. Therefore $p$ and $q$ must be relatively prime.

By definition, gluing two solid tori so that $\mu_2$ is identified with $q\lambda_1 + p\mu_1$ produces the lens space $L(q,p)$. Hence $M \cong L(q,p)$. ■

### 3.3.1. Examples

**Example 3.3.1.1**: $L(1,0) \cong S^3$

**Proof:**
For $L(1,0)$, the gluing map identifies the meridian $\mu_2$ of $V_2$ with the longitude $\lambda_1$ of $V_1$. View the handle decomposition of $V_2$. It consists of a 2-handle

$$h^2 = D^2 \times D^1$$

followed by a 3-handle.

The attaching region of $h^2$ is $\partial D^2 \times D^1$. Its central circle $\partial D^2 \times \{0\}$ is the meridian $\mu_2$, so the gluing map attaches this circle along $\lambda_1$. The core disk $D^2 \times \{0\}$ of the 2-handle therefore fills the longitudinal hole of the solid torus $V_1$. After smoothing the resulting boundary, we obtain

$$V_1 \cup h^2 \cong D^3$$

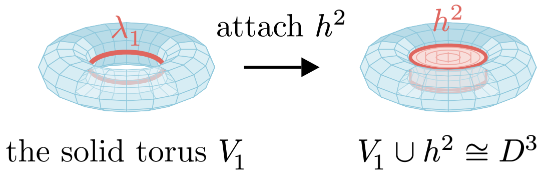


Figure 7: Attaching the 2-handle along the longitude of $V_1$ fills its central hole and produces a 3-disk.

The remaining 3-handle is also a copy of $D^3$, and it attaches along the entire boundary $S^2$ of the first 3-disk. Hence $L(1,0) \cong D^3 \cup_{S^2} D^3 \cong S^3$. ■

**Example 3.3.1.2**: $L(0,1) \cong S^2 \times S^1$

**Proof:**
Write each solid torus as $V_i = D^2 \times S^1$. Since the meridian of $V_2$ is glued to the meridian of $V_1$, we may choose the product coordinates so that the gluing identifies the boundary circle of each disk $D^2 \times \{\theta\}$ in $V_1$ with the boundary circle of the corresponding disk $D^2 \times \{\theta\}$ in $V_2$. Thus, for every $\theta \in S^1$, the two disk slices are glued together along their boundaries. Since $D^2 \cup_{S^1} D^2 \cong S^2$ we have that these pairs of disks form one copy of $S^2$ for each $\theta \in S^1$. Therefore $L(0,1) \cong (D^2 \cup_{S^1} D^2) \times S^1 \cong S^2 \times S^1$. ■

**Example 3.3.1.3**: $L(2,1) \cong \mathbb{RP}^3$

**Proof:**
Regard $\mathbb{RP}^3$ as the quotient of

$$S^3 = \left\{(z_1, z_2) \in \mathbb{C}^2 : |z_1|^2 + |z_2|^2 = 1\right\}$$

under the antipodal equivalence relation $(z_1, z_2) \sim (-z_1, -z_2)$. Decompose $S^3$ into

$$W_1 = \{(z_1, z_2) \in S^3 : |z_1| \geq |z_2|\}$$
$$W_2 = \{(z_1, z_2) \in S^3 : |z_2| \geq |z_1|\}$$

The coordinate $z_1$ never vanishes on $W_1$ as if $z_1 = 0$ then we would have $0 \geq |z_2|$ but this would imply that $|0|^2 + |0|^2 = 1$ which is contradiction. Every point of $W_1$ is therefore uniquely determined by

$$w = \frac{z_2}{z_1} \in D^2, u = \frac{z_1}{|z_1|} \in S^1$$

and conversely $w$ and $u$ determine

$$(z_1, z_2) = \frac{1}{\sqrt{1 + |w|^2}}(u, wu)$$

Thus $W_1 \cong D^2 \times S^1$. Interchanging $z_1$ and $z_2$ gives the same result for $W_2$, so both sets are solid tori. Let $\pi : S^3 \to \mathbb{RP}^3$ be the antipodal quotient map. The antipodal map preserves $W_1$ and $W_2$ because changing the signs of $z_1$ and $z_2$ does not change their absolute values. We may therefore define

$$V_1 = \pi(W_1), V_2 = \pi(W_2)$$

Under the coordinates $(w, u) \in D^2 \times S^1$ on $W_1$, the antipodal map acts by

$$(w, u) \mapsto (w, -u)$$

The quotient $S^1/(u \sim -u)$ is diffeomorphic to $S^1$ by the map $[u] \mapsto u^2$. Hence

$$V_1 \cong D^2 \times \left(S^1/(u \sim -u)\right) \cong D^2 \times S^1$$

so $V_1$ is again a solid torus. The same argument applies to $V_2$. Since $\mathbb{RP}^3 = V_1 \cup V_2$ and they meet along their common boundary, this gives a genus-one Heegaard splitting of $\mathbb{RP}^3$.

On their common boundary, write

$$z_1 = \frac{1}{\sqrt{2}}e^{i\alpha}, z_2 = \frac{1}{\sqrt{2}}e^{i\beta}$$

The antipodal identification becomes

$$(\alpha, \beta) \sim (\alpha + \pi, \beta + \pi)$$

The longitude $\lambda_1$ is obtained by fixing the disk coordinate $w$ and travelling once around the quotient circle $S^1/(u \sim -u)$. Under the diffeomorphism $[u] \mapsto u^2$, one trip around this quotient circle corresponds to changing the argument $\alpha$ of $u = \frac{z_1}{|z_2|}$ by $\pi$. Since $w = \frac{z_2}{z_1}$ remains fixed, $\beta - \alpha$ remains fixed, so $\beta$ also changes by $\pi$. Thus $\lambda_1$ has angular displacement $(\pi, \pi)$.

The meridian $\mu_1$ travels once around the boundary of the disk coordinate while the circle coordinate remains fixed. We orient it so that its angular displacement is $(0, -2\pi)$. Similarly, a meridian of $V_2$ travels once around its disk coordinate and has angular displacement $(2\pi, 0)$. Since

$$(2\pi, 0) = 2(\pi, \pi) + (0, -2\pi)$$

the meridian of $V_2$ is glued to $2\lambda_1 + \mu_1$ on the boundary of $V_1$. By the definition of a lens space, this gluing produces $L(2, 1)$. Therefore $L(2, 1) \cong \mathbb{RP}^3$. ■

The quotient description of $\mathbb{RP}^3$ above extends naturally to arbitrary lens spaces.

For the remainder of this section, we use the convention $\mathbb{Z}/0\mathbb{Z} = \mathbb{Z}$.

### 3.3.2. The Quotient Description

Regard $S^3$ as the unit sphere in $\mathbb{C}^2$,

$$S^3 = \left\{(z_1, z_2) \in \mathbb{C}^2 : |z_1|^2 + |z_2|^2 = 1\right\}$$

**Theorem 3.3.2.1**: Let $q \neq 0$ and let $p$ be relatively prime to $q$. The lens space $L(q, p)$ is diffeomorphic to the quotient of $S^3$ by the action of $\mathbb{Z}/q\mathbb{Z}$ generated by

$$(z_1, z_2) \mapsto \left(e^{2\pi \frac{i}{q}} z_1, e^{2\pi i \frac{p}{q}} z_2\right)$$

**Proof:**

Let $\zeta = e^{2\pi \frac{i}{q}}$ and let $\sigma$ denote the given transformation. Let $r$ be an integer and suppose that $\sigma^r(z_1, z_2) = (z_1, z_2)$. If $z_1 \neq 0$, then $\zeta^r = 1$, so $q$ divides $r$. If $z_1 = 0$, then $z_2 \neq 0$ and $\zeta^{pr} = 1$. Since $p$ and $q$ are relatively prime, this again implies that $q$ divides $r$. Thus no nonidentity element fixes a point and so the action is free.

For any $x \in S^3$, the points in its orbit under $\mathbb{Z}/q\mathbb{Z}$ are distinct. Since this orbit is finite, we can choose a sufficiently small neighborhood $U$ of $x$ such that the images of $U$ under the different group elements are pairwise disjoint. The quotient map therefore restricts to a diffeomorphism from $U$ onto a neighborhood of $[x]$ in $S^3/(\mathbb{Z}/q\mathbb{Z})$. These neighborhoods provide the quotient with smooth three-dimensional coordinates, so $S^3/(\mathbb{Z}/q\mathbb{Z})$ is a smooth 3-manifold.

Decompose $S^3$ into the two solid tori

$$W_1 = \{(z_1, z_2) \in S^3 : |z_1| \geq |z_2|\}$$
$$W_2 = \{(z_1, z_2) \in S^3 : |z_2| \geq |z_1|\}$$

The action preserves both sets because $|\zeta| = |\zeta^p| = 1$. Thus it does not change $|z_1|$ or $|z_2|$, so the inequalities defining $W_1$ and $W_2$ remain unchanged. To see that their quotients are solid tori, we use the coordinates

$$w_1 = \frac{z_2}{z_1} \in D^2, u_1 = \frac{z_1}{|z_1|} \in S^1$$

on $W_1$. In these coordinates, the generator acts by

$$(w_1, u_1) \mapsto (\zeta^{p-1} w_1, \zeta u_1)$$

The map

$$[w_1, u_1] \mapsto \left(w_1 u_1^{1-p}, u_1^q\right)$$

is well-defined on the quotient and is a diffeomorphism from $V_1$ to $D^2 \times S^1$. Indeed, its value is unchanged by the group action, and two points have the same image precisely when they differ by an element of the action.

For $W_2$, choose an integer $s$ such that $ps \equiv 1 \bmod q$, which exists because $p$ and $q$ are relatively prime. The element $\sigma^s$ is also a generator of the group. Using

$$w_2 = \frac{z_1}{z_2} \in D^2, u_2 = \frac{z_2}{|z_2|} \in S^1$$

it acts by $(w_2, u_2) \mapsto (\zeta^{s-1} w_2, \zeta u_2)$. The same argument, now using

$$[w_2, u_2] \mapsto (w_2 u_2^{1-s}, u_2^q)$$

shows that $V_2$ is also diffeomorphic to $D^2 \times S^1$. Thus $V_1$ and $V_2$ are solid tori.

On the common boundary, write

$$z_1 = \frac{1}{\sqrt{2}} e^{i\alpha}, z_2 = \frac{1}{\sqrt{2}} e^{i\beta}$$

The generator of the action identifies

$$(\alpha, \beta) \sim \left(\alpha + \frac{2\pi}{q}, \beta + \frac{2\pi p}{q}\right)$$

Choose the longitude $\lambda_1$ of $V_1$ to be the curve with angular displacement

$$\left(\frac{2\pi}{q}, \frac{2\pi p}{q}\right)$$

and orient its meridian $\mu_1$ so that its displacement is $(0, -2\pi)$. A meridian of $V_2$ has displacement $(2\pi, 0)$. Since

$$(2\pi, 0) = q\left(\frac{2\pi}{q}, \frac{2\pi p}{q}\right) + p(0, -2\pi)$$

the meridian of $V_2$ is glued to $q\lambda_1 + p\mu_1$ on the boundary of $V_1$. The quotient is therefore $L(q, p)$. ■

## 3.4. Classification of Lens Spaces

**Theorem 3.4.1**: The fundamental group of a Lens Space is given by

$$\pi_1(L(q, p)) \cong \mathbb{Z}/q\mathbb{Z}$$

**Proof:**
Let $L(q, p) = V_1 \cup_\varphi V_2$. Slightly thicken $V_1$ and $V_2$ across their common boundary to obtain open sets $U_1$ and $U_2$. More precisely, a neighborhood of the common boundary looks like $\mathbb{T}^2 \times (-\varepsilon, \varepsilon)$, where $\mathbb{T}^2 \times \{0\}$ is the common boundary, the negative side lies in $V_1$, and the positive side lies in $V_2$. Extend each $V_i$ a small distance across $\mathbb{T}^2 \times \{0\}$ into the other side. The resulting open sets cover $L(q, p)$, each $U_i$ deformation retracts onto $V_i$, and $U_1 \cap U_2$ deformation retracts onto $\mathbb{T}^2 \times \{0\} \cong \mathbb{T}^2$. Hence

$$\begin{aligned} \pi_1(U_1) &\cong \langle \lambda_1 \rangle \cong \mathbb{Z} \\ \pi_1(U_2) &\cong \langle \lambda_2 \rangle \cong \mathbb{Z} \\ \pi_1(U_1 \cap U_2) &\cong \langle \mu_2, \lambda_2 \mid \mu_2\lambda_2 = \lambda_2\mu_2 \rangle \cong \mathbb{Z}^2 \end{aligned}$$

The meridian $\mu_i$ bounds a disk inside $V_i$, so it is trivial in $\pi_1(U_i)$. The gluing map satisfies

$$\begin{aligned} \varphi(\mu_2) &= q\lambda_1 + p\mu_1 \\ \varphi(\lambda_2) &= r\lambda_1 + s\mu_1 \end{aligned}$$

for some integers $r$ and $s$. Under the inclusion into $U_1$, these two loops therefore represent $\lambda_1^q$ and $\lambda_1^r$, since $\mu_1$ is trivial. Under the inclusion into $U_2$, they represent the identity and $\lambda_2$, respectively. The Seifert-Van Kampen theorem now gives

$$\pi_1(L(q, p)) \cong \langle \lambda_1, \lambda_2 \mid \lambda_1^q = 1, \lambda_2 = \lambda_1^r \rangle$$

The second relation allows us to remove the generator $\lambda_2$, leaving

$$\pi_1(L(q, p)) \cong \langle \lambda_1 \mid \lambda_1^q = 1 \rangle \cong \mathbb{Z}/q\mathbb{Z}$$

■

**Theorem 3.4.2**: The lens spaces $L(q,p)$ and $L(q,p')$ are diffeomorphic if and only if

$$p' \equiv \pm p^{\pm 1} \bmod q$$

We only prove the reverse direction.

**Proof:**
Write the solid torus as $V_1 = D^2 \times S^1$ and regard its points as pairs $(z,u)$ of complex numbers satisfying $|z| \leq 1$ and $|u| = 1$. For any integer $k$, define

$$D_k(z,u) = \left(u^k z, u\right)$$

This is a diffeomorphism of the entire solid torus, with inverse $D_{-k}$. On the boundary, a meridian is obtained by varying $z$ once around $\partial D^2$ while keeping $u$ fixed, whereas a longitude is obtained by varying $u$ once around $S^1$ while keeping $z$ fixed. The map $D_k$ leaves the meridian unchanged. As $u$ travels once around $S^1$, the factor $u^k$ makes the disk coordinate travel around its boundary $k$ additional times. Hence

$$\lambda_1 \mapsto \lambda_1 + k\mu_1, \mu_1 \mapsto \mu_1$$

The restriction of $D_k$ to the boundary is called a $k$-fold Dehn twist along the meridian. The explicit formula shows that this boundary map extends over the whole solid torus.

Now suppose that $p' \equiv p \bmod q$, so $p' = p + kq$ for some integer $k$. Composing the gluing map with $D_k$ sends the gluing curve to

$$q\lambda_1 + p\mu_1 \mapsto q\lambda_1 + (p + kq)\mu_1$$

Since $D_k$ is a diffeomorphism of $V_1$, using this new gluing map produces a diffeomorphic manifold. Consequently, changing $p$ by a multiple of $q$ does not change the diffeomorphism type, and hence $L(q,p) \cong L(q,p')$.

Next consider the diffeomorphism $R : V_1 \to V_1$ given by $R(z,u) = (\overline{z}, u)$. It reflects the disk factor and leaves the circle factor fixed, so on the boundary it sends $\mu_1$ to $-\mu_1$ and preserves $\lambda_1$. Composing the gluing map with $R$ therefore changes $q\lambda_1 + p\mu_1$ into $q\lambda_1 - p\mu_1$. It follows that $L(q,p) \cong L(q,-p)$.

It remains to consider the inverse of $p$. Complete the gluing curve $q\lambda_1 + p\mu_1$ to a basis of the boundary torus. Thus, for some integers $r$ and $s$, the gluing map is represented by

$$A = \begin{pmatrix} r & q \\ s & p \end{pmatrix}$$

where the first column is the image of $\lambda_2$ and the second column is the image of $\mu_2$. Every loop on the boundary torus is represented by an integer combination of its longitude and meridian. Thus the gluing map sends $\lambda_2$ and $\mu_2$ to integer combinations of $\lambda_1$ and $\mu_1$, which gives the integer matrix $A$ above. Its inverse diffeomorphism sends $\lambda_1$ and $\mu_1$

back to integer combinations of $\lambda_2$ and $\mu_2$, giving another integer matrix $B$. Applying the gluing map and then its inverse returns every loop to itself, so

$$BA = I$$

Hence $B = A^{-1}$ has integer entries. Therefore

$$\det(A)\det(A^{-1}) = 1$$

and both factors are integers. It follows that $\det(A) = \pm 1$, so $A \in GL_2(\mathbb{Z})$.

Let $d = \det A = rp - qs$. Since $d = \pm 1$, the inverse matrix is

$$A^{-1} = d\begin{pmatrix} p & -q \\ -s & r \end{pmatrix}$$

Interchanging the two solid tori replaces the gluing map by its inverse. In this new description, the meridian being glued is $\mu_1$, so its image is given by the second column of $A^{-1}$, namely

$$\begin{pmatrix} -dq \\ dr \end{pmatrix}$$

Thus the longitude coefficient is $\pm q$ and the meridian coefficient is $\pm r$. Reversing the orientation of the gluing curve if necessary, the resulting lens space therefore has the form $L(q, \pm r)$.

The determinant equation also gives

$$rp - qs = d$$

and hence $rp \equiv d \bmod q$. Since $d = \pm 1$ and $p$ is invertible modulo $q$, we obtain

$$r \equiv \pm p^{-1} \bmod q$$

Therefore interchanging $V_1$ and $V_2$, followed if necessary by the reflection above and meridional Dehn twists, gives

$$L(q, p) \cong L(q, p')$$

whenever $p' \equiv \pm p^{\pm 1} \bmod q$. ■

## 3.5. Heegaard Splitting of the 3-Torus

The 3-torus may be written as

$$\mathbb{T}^3 = S^1 \times S^1 \times S^1 \cong \mathbb{R}^3/\mathbb{Z}^3$$

**Theorem 3.5.1**: The Heegaard genus of $\mathbb{T}^3$ is 3.

**Proof:**

Represent $\mathbb{T}^3$ by a cube with each pair of opposite faces identified. Under these identifications, all eight vertices of the cube become a single vertex. A small neighborhood of this vertex gives one 0-handle.

The twelve edges of the cube form three equivalence classes: the edges parallel to the $x$ -, $y$-, and $z$-axes. Each class becomes a loop beginning and ending at the single vertex. Thickening these three loops gives three 1-handles. The six faces form three equivalence classes, one for each pair of opposite faces, and these give three 2-handles. Finally, the interior of the cube gives one 3-handle. Thus the cube gives $\mathbb{T}^3$ a handle decomposition with one 0-handle, three 1-handles, three 2-handles, and one 3-handle.

The 0- and 1-handles form a genus-three handlebody $V_1$. Turning the decomposition upside down, the 3-handle becomes a 0-handle and the three 2-handles become 1-handles. They form another genus-three handlebody $V_2$. Hence

$$\mathbb{T}^3 = V_1 \cup_{\Sigma_3} V_2$$

so the Heegaard genus of $\mathbb{T}^3$ is at most 3.

We now show that it cannot be smaller. A genus-$g$ handlebody has a fundamental group generated by $g$ loops, one passing through each 1-handle. If a closed 3-manifold has a genus-$g$ Heegaard splitting, attaching the second handlebody adds relations to these generators but does not add any new generators. Its fundamental group can therefore be generated by at most $g$ elements.

On the other hand, $\pi_1(\mathbb{T}^3) \cong \mathbb{Z}^3$ and $\mathbb{Z}^3$ cannot be generated by fewer than three elements. Therefore every Heegaard splitting of $\mathbb{T}^3$ has genus at least 3. Combining the two inequalities, the Heegaard genus of $\mathbb{T}^3$ is exactly 3. ■